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S^3 as a cover of S^3 branched over a knot

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Abstract

We prove that for a given a torus knot $\tau_{p,q}$ in S^3 the 3-sphere is a cover of S^3 branched over $\tau_{p,q}$; we extend this result to certain family of satellites of torus knots which includes all iterated torus knots. We give some applications, namely, we find families of universal knots. © 2002 Published by Elsevier Science B.V.

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0. Introduction

A branched covering over a knot k is a covering space of the 3-sphere S^3 whose branching set is the knot k. In [2] Fox studied a certain class of irregular coverings of S^3 branched along some knot or link which turned out to be homotopy spheres, and asked if these are S^3. Burde [1] and Montesinos [7] independently showed that all the homotopy spheres constructed by Fox are S^3. In this context Montesinos asked in relation to Fox's question:

Given a knot k in S^3, is the 3-sphere a covering of S^3 branched over k?

The answer is true for universal knots (recall that a knot or link L is *universal* if every closed, orientable 3-manifold can be represented as a covering of S^3 branched over L); however for non-universal knots the answer is not clear.

In this work we answer Montesinos' question in the affirmative for torus knots and a certain class of satellites of torus knots which includes all iterated torus knots. Notice that iterated torus knots are not universal since they are fibers of a graph-manifold structure of S^3, therefore can only be the branching set of graph-manifolds (see [3]).

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Applying the technique in this work we find a new family of universal knots and we prove that any torus knot has infinitely many doubles that are universal; this result extends Theorem 1 of [6].

In Section 2 we prove the main theorems; Section 1 contains all the technical support for Section 2. Applications are given in Section 3.

As notation, $\langle g_1, g_2, \ldots, g_n\rangle$ is the subgroup generated by the group elements $g_1, g_2, \ldots, g_n$. If X is a set (which might be a group), $|X|$ will denote the cardinality of X; if X is a topological space, $|X|$ is the number of components of X; and if X is a permutation, $|X|$ is the number of orbits of X. The group of permutations on σ symbols will be denoted by S_σ.

Let $f : M \to N$ a branched covering; the points of M at which f fails to be a local homeomorphism is called the *singular set* of f and will be denoted by S_f.

1. Symmetric groups

Throughout this section we work with the symmetric group S_σ on σ elements. We need some notation and definitions.

In many cases S_σ is regarded as S_X, where S_X is the set of bijections of X and $|X| = \sigma$.

The composition of two permutations $X \xrightarrow{\alpha} X \xrightarrow{\beta} X$ is written $\alpha\beta$.

Let π be a permutation of S_σ, we say that π is a *regular permutation* of order n if π is the product of disjoint n-cycles and $|\pi| = \sigma/n$.

If π is a permutation of S_X; we define:

$$\mathit{fix}(\pi) = \{x\colon (x)\pi = x\} \quad \text{and} \quad \big(\mathit{fix}(\pi)\big)' = \{x\colon (x)\pi \neq x\}.$$

Proposition 1.1. *Let p, q be two integers, $1 \leqslant p < q$ with* $\gcd\{p, q\} = 1$. *Then there are two regular permutations α, β in S_{pq} with order p and q, respectively, such that*:

(i) $|\mathit{fix}(\beta\alpha)| \geqslant (p-1)(q-1)$.

(ii) $(\mathit{fix}(\beta\alpha))' \cap (\mathit{fix}(\beta^{-1}\alpha^{-1}))' \neq \emptyset$.

(iii) $\langle \alpha, \beta\rangle$ *is transitive.*

Proof. By induction on p. If $p = 1$ then $\alpha = \mathrm{id}$; so $|\mathit{fix}(\beta\alpha)| = 0$, and $(\mathit{fix}(\beta\alpha))' \cap (\mathit{fix}(\beta^{-1}\alpha^{-1}))' \neq \emptyset$.

Suppose $p > 1$ and let $q = np + r$, $0 < r < p$.

We identify S_{pq} with $S_{X\cup Y}$ where $X = \{1, \ldots, p\} \times \{1, \ldots, np\}$, $|Y| = rp$, and $X \cap Y = \emptyset$.

The element $(i, j) \in X$ is denoted by $x_{i\,j}$. In the following we are thinking of the elements of the sets X, Y as being entries of matrices. We define permutations β', β'' that leave the columns invariant.

Let $\beta' = \prod_{i=1}^{p}(x_{i\,1}, \ldots, x_{i\,np})$ and $\alpha' = \prod_{i=1}^{p}\prod_{j=1}^{n}(x_{i\,jp}, x_{i\,jp-1}, \ldots, x_{i\,jp-p+1})$.

Now, by induction hypothesis, there are two regular permutations α'', β'' in S_Y with order p and r, respectively, such that $|\mathit{fix}(\beta''\alpha'')| \geqslant (r-1)(p-1)$, $(\mathit{fix}(\beta''\alpha''))' \cap (\mathit{fix}(\beta''^{-1}\alpha''^{-1}))' \neq \emptyset$ and $\langle \alpha'', \beta''\rangle$ is transitive.

We can write $\beta'' = \prod_{i=1}^{p}(y_{i\,1}, \ldots, y_{i\,r})$, where

$$Y = \{y_{1\,1}, \ldots, y_{1\,r}, y_{2\,1}, \ldots, y_{2\,r}, \ldots, y_{p\,1}, \ldots, y_{p\,r}\}$$

and $y_{i\,1} \notin (\mathit{fix}(\beta''\alpha''))$; such element exists because $\langle \alpha'', \beta'' \rangle$ is transitive in Y.

Let $\delta = \prod_{i=1}^{p}(x_{i\,1}\, y_{i\,1})$.

We define $\alpha = \alpha''\alpha'$ and $\beta = \beta'\delta\beta''$. Note that α and β are regular permutations with orders p and q.

Is not difficult to see that $x_{i\,j} \in \mathit{fix}(\beta\alpha)$ if p is not a divisor of j and $y_{i\,j}$ is a fixed point of $\beta\alpha$ if $j > 1$; therefore:

$$|\mathit{fix}(\beta\alpha)| \geqslant pn(p-1) + (r-1)(p-1) = (p-1)(pn+r-1) = (p-1)(q-1).$$

To see $(\mathit{fix}(\beta\alpha))' \cap (\mathit{fix}(\beta^{-1}\alpha^{-1}))' \neq \emptyset$ we consider, first, the case $r = 1$. $y_{1\,1} \in (\mathit{fix}(\beta\alpha))' \cap (\mathit{fix}(\beta^{-1}\alpha^{-1}))'$ for

$$(y_{1\,1})\beta^{-1}\alpha^{-1} = (x_{1\,np})\alpha^{-1} = x_{1\,np-p+1}$$

and $y_{1\,1} \notin \mathit{fix}(\beta\alpha)$

Now suppose $r > 1$ and let $y_{i'\,j'} \in (\mathit{fix}(\beta''\alpha''))' \cap (\mathit{fix}(\beta''^{-1}\alpha''^{-1}))'$. If $j' = 2$ then $y_{i'\,2} \in (\mathit{fix}(\beta\alpha))' \cap (\mathit{fix}(\beta^{-1}\alpha^{-1}))'$ because

$$(y_{i'\,2})\beta^{-1}\alpha^{-1} = (x_{i'\,np})\alpha^{-1} = x_{i'\,np-p+1}.$$

If $j' \neq 2$ then $(y_{i'\,j'})\beta^{-1}\alpha^{-1} = (y_{i'\,j'})\beta''^{-1}\alpha''^{-1} \neq y_{i'\,j'}$; therefore

$$y_{i'\,j'} \in \big(\mathit{fix}(\beta\alpha)\big)' \cap \big(\mathit{fix}\big(\beta^{-1}\alpha^{-1}\big)\big)'.$$

Item (iii) follows from the induction hypothesis and the definition of β. $\square$

The technical property (ii) will be used in the next proposition.

Proposition 1.2. *Let p and q be integers, $1 < p < q$ and $\gcd\{p, q\} = 1$, let m be an integer greater than zero. Then there are two regular permutations α and β in S_{mpq} with order p and q, respectively, such that*:

(1) $|\mathit{fix}(\beta\alpha)| \geqslant m((p-1)(q-1)-1)+1$,
(2) $\langle \beta, \alpha \rangle$ *is transitive.*

Proof. We think of S_{mpq} as $S_{\mathbb{X}}$, $\mathbb{X} = \coprod_{k=1}^{m} X^k$, $|X^k| = pq$.

By Proposition 1.1, there are two regular permutations $\widehat{\alpha}$, $\widehat{\beta} \in S_{\mathbb{X}}$ with order p and q such that the orbits of $\langle \widehat{\alpha}, \widehat{\beta} \rangle$ are $X^1, \ldots, X^k$, $|\mathit{fix}(\widehat{\beta}|_{X^k}\widehat{\alpha}|_{X^k})| > 0$ and $(\mathit{fix}(\widehat{\beta}|_{X^k}\widehat{\alpha}|_{X^k})' \cap (\mathit{fix}(\widehat{\beta}^{-1}|_{X^k}\widehat{\alpha}^{-1}|_{X^k}))' \neq \emptyset$.

We write

$$X^k = \big\{x^k_{i\,j}\colon 1 \leqslant i < p,\ 1 \leqslant j \leqslant q\big\} \quad \text{and} \quad \widehat{\beta} = \prod_{\substack{1 \leqslant k \leqslant m\\ 1 \leqslant i \leqslant p}} \big(x^k_{i\,1}, \ldots, x^k_{i\,q}\big),$$

where $(x^k_{1\,1})\widehat{\beta}\widehat{\alpha} = (x^k_{1\,1})$ if $1 \leqslant k < m$ and

$$\big(x^m_{1\,1}\big) \in \big(\mathit{fix}\big(\widehat{\beta}\widehat{\alpha}|_{X^m}\big)\big)' \cap \big(\mathit{fix}\big(\widehat{\beta}^{-1}\widehat{\alpha}^{-1}|_{X^m}\big)\big)' \neq \emptyset.$$

We define

$$\beta = \prod_{1\leqslant k\leqslant m} \left(x_{1\,1}^k, x_{1\,2}^{k+1}, \ldots, x_{1\,q}^{k+1}\right) \prod_{\substack{1\leqslant k\leqslant m\\ 1<i\leqslant p}} \left(x_{i\,1}^k, \ldots, x_{i\,q}^k\right),$$

where $k+1=1$ if $k=m$. Let $\alpha = \widehat{\alpha}$.

Now, (1) follows from these two facts:

(a) $(z)\widehat{\beta}\widehat{\alpha} = (z)$ if $(z)\beta\alpha = (z)$ unless $z = x_{1\,1}^k$, $1 \leqslant k < m$.

(b) $|\mathit{fix}(\beta\alpha|_{X^m})| \geqslant (p-1)(q-1)$.

(a) is clear from the definitions of β and α.

To see (b) notice that $\mathit{fix}(\beta\alpha|_{X^m}) = \mathit{fix}(\widehat{\beta}\widehat{\alpha}|_{X^m})$, for $(x_{i\,j}^m)\beta\alpha = (x_{i\,j}^m)\widehat{\beta}\widehat{\alpha}$ if $i \geqslant 1$ and $j \notin \{1, r\}$; and $x_{1\,1}^m \in (\mathit{fix}(\beta\alpha|_{X^m}))' \cap (\mathit{fix}(\widehat{\beta}\widehat{\alpha}|_{X^m}))'$.

Property (2) follows from: If $j > 1$ then $(x_{1\,j}^k)\beta^s = (x_{1\,2}^k)$, $s \in \mathbb{Z}$; also $(x_{1\,2}^k)\alpha = (x_{1\,2}^k)\widehat{\alpha}\widehat{\beta} = (x_{1\,1}^k)$ if $1 \leqslant k < m$; moreover $(x_{1\,1}^k)\beta = (x_{1\,2}^{k+1})$. Therefore, $x_{1\,j}^k$ is in the orbit of $\langle\alpha, \beta\rangle$ for any $x_{1\,j}^k$. Now, note that there is a permutation $\rho \in \langle\alpha, \beta\rangle$ such that $(x_{i\,j}^k)\rho = x_{1\,j}^k$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$. So $\langle\alpha, \beta\rangle$ is transitive in S_{mpq}. $\square$

Example 1.3. Let $q=5$, $p=3$, $m=3$:

$$\begin{aligned}
&5 = 3(1)+2; \quad 3 = 2(1)+1,\\
&\widehat{\beta} = (1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20)\\
&\qquad (21,22,23,24,25)(26,27,28,29,30)(31,32,33,34,35)(36,37,38,39,40)\\
&\qquad (41,42,43,44,45),\\
&\beta = (1,17,18,19,20)(16,45,41,42,43)(44,2,3,4,5)(6,7,8,9,10)\\
&\qquad (11,12,13,14,15)(21,22,23,24,25)(26,27,28,29,30)(36,37,38,39,40)\\
&\qquad (31,32,33,34,35),\\
&\alpha = (3,2,1)(5,4,14)(8,7,6)(10,9,15)(13,12,11)(18,17,16)(20,19,29)\\
&\qquad (23,22,21)(25,24,30)(28,27,26)(33,32,31)(35,34,44)(38,37,36)(40,39,45)\\
&\qquad (43,42,41),\\
&\mathit{fix}(\beta\alpha)\\
&\quad = \{2,4,6,7,9,11,12,17,19,21,22,24,26,27,31,32,34,36,37,39,41,42\}.
\end{aligned}$$

2. Main theorems

In this section, we will think of $\tau_{p,q}$ as a regular fiber of the Seifert bundle $S^3 = (0,0; r/p, s/q)$, where $-r/p - s/q = \pm 1/pq$.

In the following theorem we consider the lens space $L(m,1)$ as the Seifert bundle $(0,0; \pm m/1)$. S_{f_1} denotes the singular set of f_1.

Theorem 2.1. *Let $\tau_{p,q}$ be the (p,q)-torus knot. Then, for every $m \in \mathbb{N}$, there is an mpq-fold covering $f_1 : L(m,1) \to (S^3, \tau_{p,q})$ branched along $\tau_{p,q}$ such that S_{f_1} is a fiber and $f_1|_{S_{f_1}} : S_{f_1} \to \tau_{p,q}$ is a homeomorphism.*

Proof. Let S be the base space of $(0,0; r/p, s/q)$ obtained by identifying each fiber to a point, and let $\rho : (0,0; r/p, s/q) = (S^3, \tau_{p,q}) \to S$ be the natural projection onto the base space. The restriction $\rho_0 : (S^3 - \tau_{p,q}) \to (S - y)$ of ρ $(y = \rho(\tau_{p,q}))$ induces an epimorphism ρ_{0*} of $\pi_1(S^3 - \tau_{p,q})$ onto the orbifold fundamental group $\pi_1^o(S - y) = \mathbb{Z}_p * \mathbb{Z}_q = \|z_1, z_2\colon z_1^p = 1 = z_2^q\|$ with kernel the infinite cyclic group generated by a regular fiber h.

In terms of the more usual presentation $\|a_1, a_2\colon a_1^p = a_2^q\|$ of $\pi_1(S^3 - \tau_{p,q})$ one can write $\rho_{0*}(a_1) = z_1^q$, $\rho_{0*}(a_2) = z_2^p$, $\rho_{0*}(\mu) = z_1 z_2$ where μ is the meridian $\mu = a_2^{-s} a_1^{-r}$.

Now, by Proposition 1.2 we have the following transitive representation:

$$\omega : \pi_1^o(S - y) \to S_{mpq} \qquad \text{such that } \omega(z_1) = \alpha, \omega(z_2) = \beta$$

where α is a regular permutation of order p and β is a regular permutation of order q; therefore the composition $\omega(\rho_{0*})$ is a transitive representation of $\pi_1(S^3 - \tau_{p,q})$ into S_{mpq} (note that $\omega(\rho_{0*}(h)) = \mathrm{id}$ since $h = a_1^p = a_2^q$).

Let $f_1 : \overline{M} \to (S^3, \tau_{p,q})$ be the branched covering associated to $\omega(\rho_{0*})$. The genus $\overline{g}$ of the base space $\overline{S}$ of $\overline{M}$ is given by the following formula [4] $2\overline{g} - 2 = mpq - |\omega(z_1)| - |\omega(z_2)| - |\omega(z_1 z_2)| = mpq - mq - mp - |\omega(z_1 z_2)|$.

Since $|\omega(z_1 z_2)| > m((p-1)(q-1) - 1) + 1$ and $\overline{g} \geqslant 0$ we have

$$\left|\omega(z_1 z_2)\right| = m\big((p-1)(q-1) - 1\big) + 2.$$

Therefore $\overline{g} = 0$.

Let s_p and s_q be the exceptional fibers of order p and q in $(0,0; r/p, s/q)$ and let x_p and x_q be the cone points of S that correspond to s_p and s_q. Then, $f_1^{-1}(x_p)$ and $f_1^{-1}(x_q)$ have branching indices given by the cardinalities of the orbits of $\omega(z_i)$ $(i = 1, 2)$. Fig. 1 shows how the boundary of a neighborhood of s_p and s_q lifts:

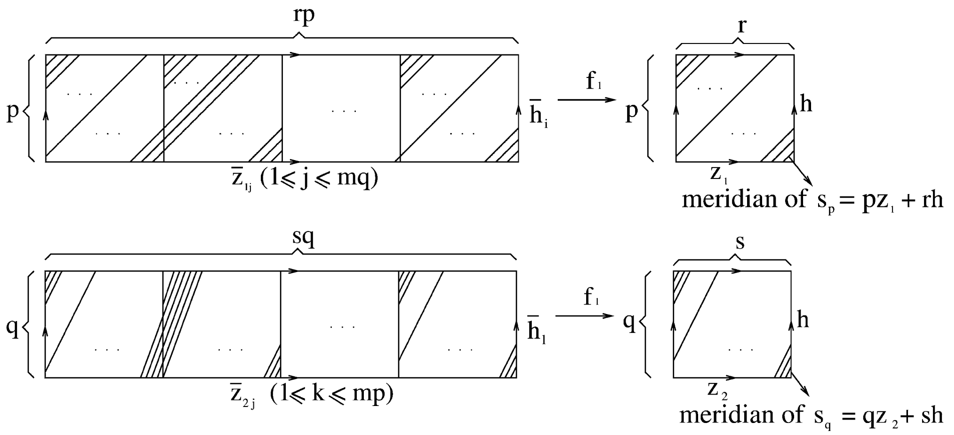

Fig. 1.

Notice that the meridian of s_p lifts to $\overline{z}_{1j} + r\overline{h}_i$, $1 \leqslant j, i \leqslant mq$; and the meridian of s_q lifts to $\overline{z}_{2k} + q\overline{h}_l$, $1 \leqslant k, l \leqslant mp$.

So $\overline{M}$ is the Seifert bundle:

$$(0,0;\underbrace{r/1,\ldots,r/1}_{mq \text{ times}},\underbrace{s/1,\ldots,s/1}_{mp \text{ times}}).$$

Is not difficult to see:

$$(0,0;\underbrace{r/1,\ldots,r/1}_{mq \text{ times}},\underbrace{s/1,\ldots,s/1}_{mp \text{ times}}) = \bigl(0,0;m(rq+ps)/1\bigr) = (0,0;\pm m/1)$$

(the sign depends on the relation $-r/p - r/q = \pm 1/pq$).

Therefore $\overline{M} = (0,0;\pm m/1) = L(m,1)$. $\square$

Corollary 2.2. *Let $\tau_{p,q}$ be the (p,q)-torus knot in S^3. Then, for any $m \in \mathbb{N}$, there is an m^2pq-fold covering $f : S^3 \to (S^3, \tau_{p,q})$ branched over $\tau_{p,q}$. Moreover, S_f is a trivial knot and $f|_{S_f} : S_f \to \tau_{p,q}$ is an m-fold covering.*

Proof. Let $f_2 : S^3 = (0,0;\pm 1/1) \to (0,0;\pm m/1) = L(m,1)$ be the universal covering of the lens space $L(m,1)$. Let $f = f_1 \circ f_2$.

The fact that $f|_{S_f}$ is m to 1 follows from this: $\pi_1(0,0;\pm m/1)$ is generated by the fiber S_{f_1}. $\square$

Note that the covering space S^3 has the Hopf fibration.

The following theorem and its corollary extend the last corollary for knots which are monotone satellites of torus knots; we need the next definitions.

Let J be a circle in $S^1 \times D^2$ such that J is not contained in a 3-ball of $S^1 \times D^2$; we say that J intersects transversally a disc $\{z\} \times D^2$ if

$$\bigl(D^2 \times I, J \cap \bigl(D^2 \times I\bigr)\bigr) \cong \bigl(D^2 \times I, F \times I\bigr),$$

where $F \subseteq D^2$ and F is a finite set.

Suppose J is a curve in $S^1 \times D^2$ as above such that J is not the core of $S^1 \times D^2$, we say that J is a *monotone curve* in $S^1 \times D^2$ if J intersects transversally $\{z\} \times D^2$ for every $z \in S^1$.

Let k be a nontrivial knot in S^3, $\eta(k)$ be a tubular neighborhood of k and $f : S^1 \times D^2 \to \eta(k)$ be a homeomorphism such that $f(S^1 \times \{z\}) = 0 \in H_1(S^3 - \mathit{int}\,\eta(k))$, $z \in \partial D^2$. For J a monotone curve in $S^1 \times D^2$, $f(J)$ is a *monotone satellite* of k; we denote it by $J(k)$.

Theorem 2.3. *Let $f : M^3 \to S^3$ be a covering branched along the knot k such that S_f is connected and $f|_{S_f}$ is m to 1. Let $J(k)$ be a monotone satellite of k whose winding number d divides m. Then there is a covering $g : M^3 \to S^3$ branched over $J(k)$ which coincides with f outside a tubular neighborhood of S_f.*

Proof. Let T be a tubular neighborhood of k containing $J(k)$ in its interior with $\pi|_{J(k)}$ a d-fold covering, where $\pi : T \to k$ is the projection. Let $\overline{\overline{T}}$ be the component of $f^{-1}(T)$

containing S_f. Then $f|_{\overline{\overline{T}}}$ is a covering of T branched along k and factors as $\overline{\overline{T}} \xrightarrow{p} \overline{T} \xrightarrow{u} T$ where u is an m-fold unbranched covering and p is a primitive covering of $\overline{T}$ branched along a core $\overline{C}$ of $\overline{T}$, that is, $p_*:\pi_1(\overline{\overline{T}}) \to \pi_1(\overline{T})$ is surjective.

Now, $u^{-1}(J(k))$ is a link with d components such that each one of them is isotopic to $\overline{C}$, for u is an m-cyclic covering and $J(k)$ is a monotone satellite of k whose winding number d divides m.

Let $h:\overline{T} \to \overline{T}$ be the homeomorphism such that h maps one of the components onto the core and $h|_{\partial\overline{T}} = \mathrm{id}$.

Define g to be f on $M^3 - \overline{\overline{T}}$ and $u \circ h^{-1} \circ p$ on $\overline{\overline{T}}$. This is the required covering. $\square$

Corollary 2.4. *The* 3*-sphere is an* m^2pq*-fold cover of* S^3 *branched over* $J(\tau_{p,q})$*, where* $\tau_{p,q}$ *is the* (p,q)*-torus knot, and the winding number of* $J(\tau_{p,q})$ *is a divisor* d *of* m*.*

Corollary 2.5. *Let* J *be an iterated torus knot. Then, the* 3*-sphere is a covering of* S^3 *branched over* J*.*

3. Applications

Now we show how to apply these results to find a new family of universal knots.

In the following we suppose that S^3, $\tau_{p,q}$ and $S^1 \times D^2$ are oriented, and $-r/p - s/q = -1/pq$.

Let J be the monotone curve in $S^1 \times D^2$ of Fig. 2 and $f:S^1 \times D^2 \to \eta(\tau_{p,q})$ a homeomorphism such that $f(S^1 \times \{z\})$ is the regular fiber of a fibration of $S^3 - \tau_{p,q}$, $z \in \partial D^2$; suppose that f is orientation preserving.

Let $\rho = u \circ p:S^1 \times D^2 \to S^1 \times D^2$ be the covering with $p:S^1 \times D^2 \to S^1 \times D^2$ a homeomorphism and $u:S^1 \times D^2 \to S^1 \times D^2$ a 3-fold covering; we denote $J_2 = (u \circ p)^{-1}(J)$.

Let $J(\tau_{p,q})$ be a monotone satellite of $\tau_{p,q}$, where J is the curve in Fig. 3. By Corollary 2.2 we have that there is a 3^2pq-fold covering $g:S^3 \to S^3$ branched over $J(\tau_{p,q})$. It is not difficult to see that $J_2(l_2)$, a lifting of $J(\tau_{p,q})$ in S^3 with branching index one, is the link in Fig. 4. But $J_2(l_2)$ is isotopic to L (see Fig. 5).

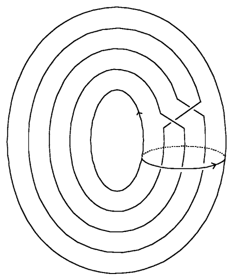

Fig. 2.

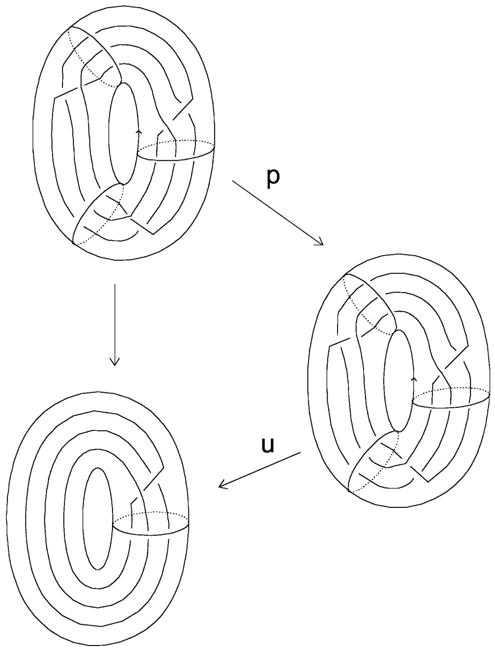

Fig. 3.

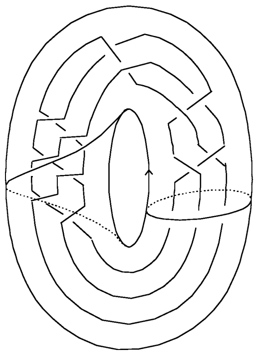

Fig. 4.

Now we consider the 2-fold covering associated to $\omega:\pi_1(S^3 - k) \to S_2$ where k is the component of $J_2(l_2)$ containing the point at infinity and ω is the only surjective representation. Then the lifting of L in S^3 is the rational link $L(-8/3)$ which is universal [5]. See Fig. 6.

We conclude:

Proposition 3.1. *Every (p,q)-torus knot $\tau_{p,q}$ has a monotone satellite with winding number* 3 *which is universal.*

As another application we have:

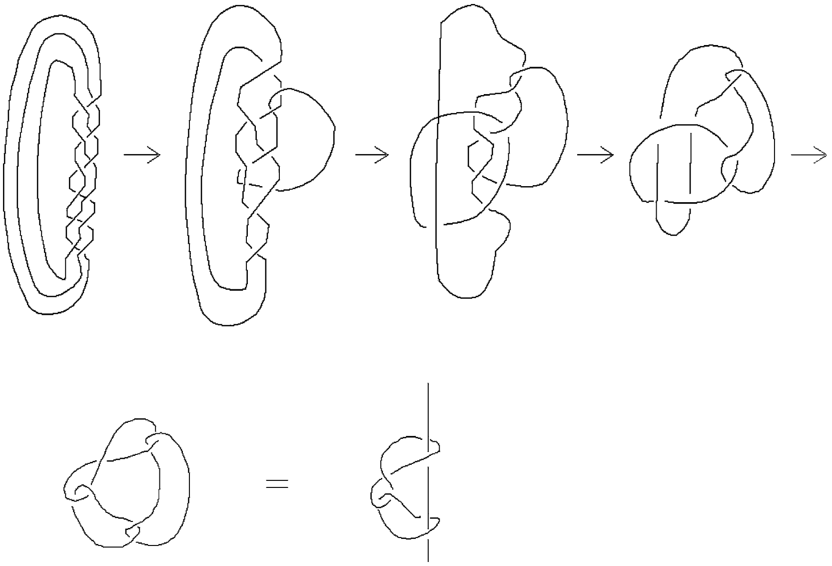

Fig. 5.

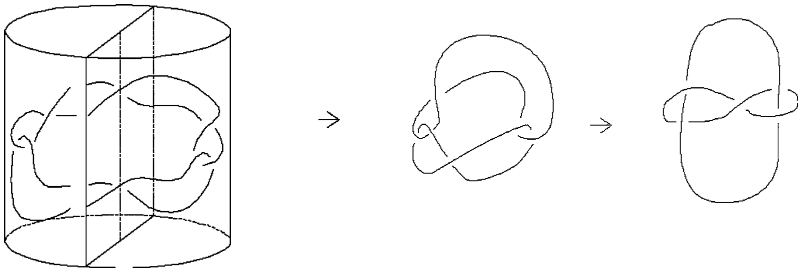

Fig. 6.

Theorem 3.2. *Let $\tau_{p,q}$ be the (p,q)-torus knot in S^3. Then infinitely many doubles of $\tau_{p,q}$ are universal.*

Proof. By Corollary 2.2 there is a 2^2pq-fold covering $f: S^3 \to (S^3, \tau_{p,q})$ branched over $\tau_{p,q}$ whose singular set, S_f, is connected and $f|_{S_f}$ is a 2-fold covering.

Let T be a tubular neighborhood of $\tau_{p,q}$ containing $\tau'_{p,q}$ in its interior ($\tau'_{p,q}$ is another fiber of the Seifert bundle $(S^3, \tau_{p,q})$). Let $\overline{\overline{T}}$ be the component of $f^{-1}(T)$ containing S_f; then $f|_{\overline{\overline{T}}}$ is a covering of T branched over $\tau_{p,q}$, and factors as $\overline{\overline{T}} \xrightarrow{p} \overline{T} \xrightarrow{u} T$ where u is a 2-fold unbranched covering and p is a $(2(p+q)-1)$-fold covering of $\overline{T}$ branched over $u^{-1}(\tau_{p,q})$.

We can replace p by $\widehat{p}: \overline{\overline{T}} \to T$, the $(2(p+q)-1)$-fold irregular dihedral covering branched over the 2-cable $\overline{L} = u^{-1}(\tau_{p,q}) \cup u^{-1}(\tau'_{p,q})$, to obtain $\hat{f} = u(\widehat{p})$ a new $(2(2(p+q)-1))$-fold covering over $(S^3, \tau_{p,q} \cup \tau'_{p,q})$; in fact $\widehat{p}$ is the covering associated to the representation $\widehat{\omega}: \pi_1(\overline{T} - \overline{L}) \to S_{2(p+q)-1}$ defined as:

$$\widehat{\omega}(m_1) = (1\ 2)(3\ 4)\cdots(2(p+q)-3\ 2(p+q)-2),$$
$$\widehat{\omega}(m_2) = (2\ 3)(4\ 5)\cdots(2(p+q)-2\ 2(p+q)-1),$$
$$\widehat{\omega}(\overline{h}) = \mathrm{id},$$

where m_1, m_2 are the meridians corresponding to $u^{-1}(\tau_{p,q})$ and $u^{-1}(\tau'_{p,q})$, respectively, and $\overline{h}$ is the preimage of a fiber $h \subset \partial T$, $\overline{h} = u(h)$.

Let $g : S^3 \to S^3$ be the map f on $S^3 - \overline{\overline{\overline{T}}}$ and the map $\hat{f}$ on $\overline{\overline{\overline{T}}}$. Now we modify $L = \tau_{p,q} \cup \tau'_{p,q}$ as in Fig. 7. Notice that this move changes the link but does not change the manifold upstairs, for the preimages of the 3-ball B^3 in Fig. 8 under u are two 3-balls B_i^3 $(i = 1, 2)$ and $\widehat{p}^{-1}(B_i^3)$ is again a 3-ball [5].

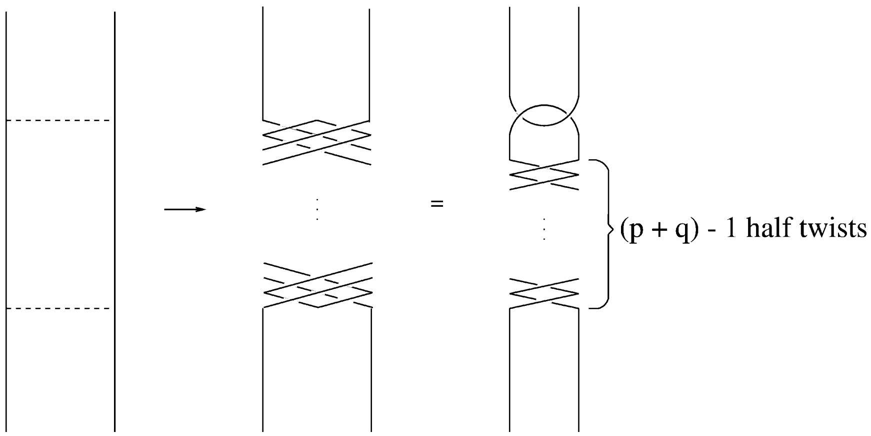

Fig. 7.

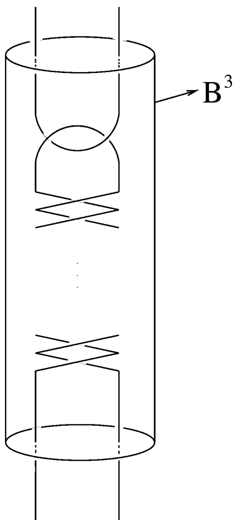

Fig. 8.

In a similar way we can make n moves as shown in Fig. 9.

Now, to prove that the doubles obtained by these moves (one of the first kind and n moves of the second kind) are universal we note that any component of branching index one of the preimage of L is the link in Fig. 10 else (the new single twist appears because $(S^3, f^{-1}(\tau_{p,q}))$ has the Hopf fibration).

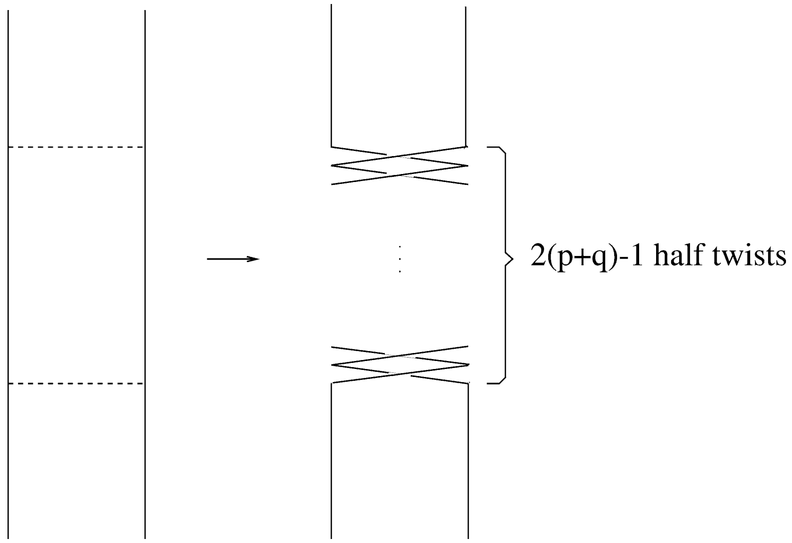

Fig. 9.

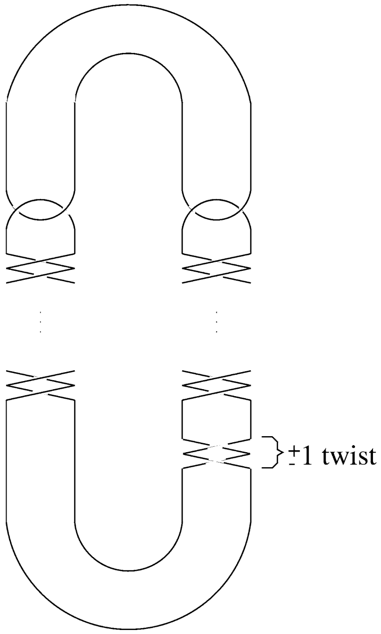

Fig. 10.

But it is known these 2-bridge links are universal unless the total sum of half twists vanishes, but there are a finite number of choices [5]. □

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